

# MHD of rotating compact stars with spectral methods: description of the algorithm and tests

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## Abstract

A flexible spectral code for the study of general relativistic magnetohydrodynamics is presented. Aiming at investigating the physics of slowly rotating magnetized compact stars, this new code makes use of various physically motivated approximations. Among them, the relativistic anelastic approximation is a key ingredient of the current version of the code. In this article, we mainly outline the method, putting emphasis on algorithmic techniques that enable to benefit as much as possible of the non-dissipative character of spectral methods, showing also a potential astrophysical application and providing a few illustrative tests.

## 1 Introduction and motivation

During the last decades, Astrophysics has seen fast progress in its observational techniques, and a very wide flow of data is now coming from several orbital and ground-based detectors. This huge amount of information is a challenge not only to the data analysis community, but also to theorists, as accurate numerical models become more and more required to explain various astrophysical observations and to put to the test more qualitative theoretical predictions. Among the processes that are the most interesting and complicated from both physical and numerical points of view, those related with strong gravitational and/or magnetic fields play crucial roles inside or in the vicinity of compact objects. Hence, they are involved in the description of numerous phenomena, such as are massive stars core collapses, emergence of jets from active galaxies, emission of gamma-ray bursts (GRB) or of gravitational waves (GW), merger of binary neutron stars (NS) or just evolution of isolated NS. As a consequence, various codes for relativistic Magneto-HydroDynamics (MHD) have been developed [e.g. Koide *et al.* (1999), De Villiers & Hawley (2003), Gammie *et al.* (2003), Komissarov (2004), Shibata & Sekiguchi (2005), Duez *et al.* (2005), Antón *et al.* (2006), Mizuno *et al.* (2006)], with different scientific purposes, and therefore distinct choices in the physical approximations, in the writing of the equations and in the numerical implementation.

In the following, we present a new relativistic MHD code, mainly describing the methods and some basic tests. This code is tridimensional

and based on spectral methods, quite similar to the relativistic hydrodynamical code introduced in Villain & Bonazzola (2002) (hereafter referred to as Paper I) and Villain, Bonazzola & Haensel (2005) (Paper II). The new code was designed to be flexible and to fulfill several demands, the first of them being to solve MHD equations in the fixed curved space-time associated with the steady-state configuration of a slowly rotating NS (*i.e.* it currently works in the Cowling approximation). Those equations are solved in the whole interior of the star (including the origin), using *spherical vector components* and *spherical coordinates*, which is almost necessary to easily implement various boundary conditions (BC), and hence to be able to deal with a wide spectrum of physical situations. Another requirement for exploring numerous physical phenomena is that of the code being robust and stable in the sense of allowing to follow the evolutionary tracks during many rotational periods. In such a way, it can also be used to study the stability of equilibrium configurations of rotating magnetized NSs. As we shall illustrate, this is made possible by the non-dissipative character of spectral methods and by some cautions taken in the algorithm, such as having the numerical magnetic field that remains divergence-free.

The article is divided in several sections: in Sect. 2, we summarized the typical orders of magnitude and timescales involved in the physical situations that we shall study in the future, using the resulting approximations to write the relativistic equations of motion in a Newtonian-like way. Then, the numerical method is described in Sect. 3. We discuss some possible applications in Sect. 4. Finally, Sect. 5 contains tests performed, whereas final conclusions are gathered in Sect. 6.

## 2 Timescales and equations of motion

We shall briefly remind orders of magnitude and timescales for the type of problems to be addressed with the code: We are interested in the MHD of old but not too cold (not superfluid) compact stars, which have radii of some  $10^6$  cm, mean baryonic densities  $n \sim 10^{14}$  g/cm<sup>3</sup>, and magnetic fields of  $10^{12}$  to  $10^{16}$  gauss. We consider only compact stars whose density does not vanish at the surface (the so-called bare strange stars) or which have a rigid crust. In the latter case the domain of integration is limited to the liquid part of the star, completely liquid stars (just born very hot NSs) being much more difficult to treat since BC of wind solutions have to be imposed<sup>1</sup>.

Most of known isolated rotating NSs do not have rotational frequency larger than 300 Hz ( $\sim \Omega = 1880$  rad/s), which gives a period  $\tau_{rot} = 3.3 \cdot 10^{-3}$  s. Retaining this typical value implies that to keep only terms linear in  $\Omega$  is a good approximation, be it for the relativistic Euler equation or the background metric. Using also the conformal flatness approximation, the latter is written as in Papers I and II in a  $(3 + 1)$ -like isotropic way (in  $c = 1$  units)

$$ds^2 = -N^2 dt^2 - 2r^2 \sin(\theta)^2 a^2 N_3 d\phi dt + a^2 dl^2, \quad (1)$$

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<sup>1</sup>This kind of problem can be treated with two different numerical techniques: A finite difference method [like the one developed by the Valencia group, see for instance Antón *et al.* (2006)] during the in-falling and cooling phase, then the present algorithm once the crust is formed. On NSs winds, see also Bucciantini *et al.* (2006).

where  $dl^2$  is the Euclidean infinitesimal 3-length, while  $r$  and  $\theta$  are spherical coordinates, with the lapse  $N$ , the conformal factor  $a$ , and the shift  $N_3$  that only depend on  $r$ .

Another important velocity for magnetized NSs is the Alfvén velocity defined as  $c_a \sim B/\sqrt{4\pi n} \sim 10^4 B_{12}$  cm/s [with  $B_{12} = B/(10^{12}\text{gauss})$ ], leading to the Alfvén crossing time  $\tau_{Al} = 10^2 B_{12}^{-1}$  s (and to  $\nu_{Al} = 10^{-2} B_{12}$  Hz). This time and the period are much larger than the acoustic crossing time  $\tau_{ac}$ , whose value comes from the sound velocity that, in compact stars, is approximately equal to one third of the light velocity ( $c_s \simeq 0.3c \rightarrow \tau_{ac} \sim 3.3 \cdot 10^{-5}$  s, with corresponding frequency:  $\nu_{ac} \sim 3 \cdot 10^5$  Hz). As  $\tau_{ac}$  is much shorter than the other characteristic times, we shall filter acoustic waves through the so-called relativistic anelastic approximation introduced in Paper I:

$$\frac{1}{\sqrt{e}} \partial_i (\sqrt{e} \mathcal{P}^i) = 0, \quad (2)$$

with  $e$  the determinant of the Euclidean 3-metric and  $\mathcal{P}^i \equiv n N a^3 U^i$ , where  $U^i$  is the spatial part of the fluid 4-velocity ( $i \in [1, 3]$ ). Notice that contrary to what was done in Papers I and II, this relation is not necessarily linearized here, for we shall also use the divergence-free character of the  $\mathcal{P}^i$  3-vector as a key feature for the algorithmic (see Sect. 3). Moreover, we shall already mention that the momentum density  $\mathcal{P}$  (which is the 1-form associated to  $\vec{\mathcal{P}}$ ) is the main variable retained in the hydrodynamical part of the new code, while in previous works (Papers I and II), this variable was the velocity  $\vec{V}$ . The reason of the previous choice was that the vanishing matter density at the surface of the star (at least in Paper I), would not have made possible in general to recover the velocity from the momentum density<sup>2</sup>.

As the star can be described as composed of a single neutral fluid (see Paper II), we follow Wilson (1972) and use both the form of the metric (1) and the anelastic approximation (2), to write Euler equation in the Newtonian-like way

$$\begin{aligned} \partial_t \mathcal{P}_i + \frac{nN}{f} \vec{\nabla}_j \left( \frac{f}{n} U^j \mathcal{P}_i \right) + \frac{a n N^2}{f} \vec{\nabla}_i P = \\ \frac{\mathcal{P}_\mu \mathcal{P}_\nu}{2 a^2 N \mathcal{P}^0} \vec{\nabla}_i g_{\mu\nu} + \frac{a^3 N^2 n}{f} \mathcal{F}_{i\mu} \mathcal{J}^\mu, \end{aligned} \quad (3)$$

where  $f \equiv \rho + P$ , whereas  $\mathcal{F}_{i\mu} \mathcal{J}^\mu$  is the Lorentz force.

Mainly due to the frame-dragging effect, to curvature of space-time and to the possible non-ideality of the plasma, the Lorentz force contains numerous terms that we shall not all make explicit here. We shall just write it in such a way to make more visible the equivalent of the Newtonian force, which will be used in Sect. 5 to proceed to basic tests of the code

$$\mathcal{F}_{i\mu} \mathcal{J}^\mu \sim \frac{Z}{4\pi} (\vec{\nabla} \wedge \vec{B}) \wedge \vec{B} + RT, \quad (4)$$

where  $Z$  is a metric factor,  $\vec{B}$  a magnetic field-like vector and  $RT$  encompasses all other Relativistic Terms (related with frame-dragging, non-

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<sup>2</sup>The division of  $\vec{\mathcal{P}}$  by the matter density  $n$  is numerically possible only for stiff equation of state ( $\Gamma \geq 2$ ).

ideality of the conductivity, *etc.*).  $\vec{B}$  is defined as

$$B^i = \frac{1}{2} \eta^{ijk} F_{jk}, \quad (5)$$

with  $\eta^{ijk}$  the tridimensional Euclidean Levi-Civita volume form and  $F_{jk}$  the Maxwell electromagnetic tensor. Its evolution is ruled by the so-called induction equation, which comes from relativistic Ohm's law<sup>3</sup> applied to a neutral plasma

$$\kappa J_\mu = F_{\mu\nu} U^\nu, \quad (6)$$

where  $\kappa$  is the conductivity<sup>4</sup>, and from the homogeneous Maxwell equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \quad (7)$$

The relativistic induction equation for  $\vec{B}$  is written in a Newtonian-like way as

$$\partial_t \vec{B} + \vec{\nabla} \wedge (\vec{B} \wedge \vec{V}) + \vec{\nabla} \wedge (\kappa \vec{\mathcal{J}}) = 0, \quad (8)$$

where  $V^i = U^i/U^0$  and  $\mathcal{J}^i = N J^i$ . Notice that due to the frame-dragging effect,  $J^i$  is not directly related to the curl of  $\vec{B}$  by the inhomogeneous Maxwell equation, but we shall not enter more into the detail here, and we just mention that while testing the code, we used the Newtonian expression of Faraday's equation in the MHD approximation

$$\frac{1}{4\pi} \vec{\mathcal{J}} = \vec{\nabla} \wedge \vec{B}. \quad (9)$$

The main reason behind that approximation is that it does not change the algorithmic but allows to verify the conservation of energy (see Sect.5) and consequently the whole mechanics of the code and its non-dissipative character. Furthermore, we would like to insist on the fact that what we present here is the simplest version of the code, containing the minimal number of physical assumptions. Thanks to the required flexibility, more physical situations can be easily considered (for example strong stratification, see further), by following the same strategy as going from Paper I to Paper II. However, notice that when treating non-linear problems, we will need some kind of artificial viscosity in order to smooth the solutions. Yet, the main difference between spectral and finite difference scheme codes is that with spectral codes, artificial viscosity and resistivity are put as differential operators, implying that we perfectly know their behaviour, while they also help to impose boundary conditions. Since for linearized<sup>5</sup> problems, we do not need, in principle, dissipative terms, specific problems can arise due to the fact that instabilities have much more possibilities to develop. Consequently, the linearized non-dissipative version of the code is still in progress, even if some tests are already provided here (see Sect. 4), showing that the global strategy works well.

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<sup>3</sup>See for instance Lichnerowicz (1967).

<sup>4</sup>Notice that the decay time of magnetic field in NS is, as the viscous time, much larger than the dynamical time under consideration here. See for instance Goldreich & Reisenegger (1992).

<sup>5</sup>Notice that what we call *the linear case* is a linearization around the rigid rotation solution, implying equations of motion very similar to the ones described in Paper I and II.

### 3 The method

#### 3.1 Implementation of the anelastic approximation

As in Papers I and II, the solving of the equations (linear or not) relies on two main ingredients:

- the Helmholtz theorem, that is to say the decomposition of vectors into divergence-free components and potential parts,
- the use of so-called angular potentials.

However, working with as variable the momentum makes the implementation of the anelastic approximation quite different, and we proceed by considering the equation of motion (Eq. 3) written in the following way:

$$\partial_t \mathcal{P}_i + \partial_i \tilde{P} = S_i, \quad (10)$$

where  $\tilde{P}$  is a kind of geometrically modified pressure ( $\tilde{P} \equiv a N^2 \Pi$  with  $d\Pi \equiv ndP/f$ ), while the term  $S_i$  represents all other terms appearing in Eq. (3) (non-linear included) and which are considered as a known source. Applied to this known source term  $S_i$ , Helmholtz theorem gives a divergence-free component  $\tilde{S}_i$  and a potential part  $\partial_i \Psi$ :

$$S_i = \tilde{S}_i + \partial_i \Psi. \quad (11)$$

Note that the above decomposition is not unique, since one can add an arbitrary harmonic function  $\Psi_A$  to the potential  $\Psi$  (or equivalently subtract its gradient to  $\tilde{S}_i$ ) without altering the divergence of  $S_i$ . As we shall see in the following, this degree of freedom is used to impose an arbitrary BC for  $\tilde{P}$  (for example  $\mathcal{P}_r = 0$ ). Due to the anelastic approximation and the exact compensation that is required between  $\partial_i \tilde{P}$  and  $\partial_i \Psi$ , the equation to be solved reads

$$\partial_t \mathcal{P}_i = \tilde{S}_i. \quad (12)$$

If the non-linear equations are considered, a viscosity  $\zeta$  (supposed to be constant) is added, and we have

$$\partial_t \mathcal{P}_i + \zeta \Delta \mathcal{P}_i = \tilde{S}_i \quad (13)$$

Once  $\tilde{P}$  is obtained ( $\tilde{P} = \Psi$ ), the new source term for the next time step can be calculated (in the barotropic case) by using the equation of state and the frozen metric. But before entering more into the detail of solving such equations (the induction equation being technically identical), we shall now briefly recap on angular potentials and their use in spherical geometry.

#### 3.2 Vector angular potentials

As mentioned in Sect. 1, we are interested in solving vectorial partial differential equations (PDE) employing spherical components and spherical coordinates. The advantage of this choice is the possibility to impose exact boundary conditions at the surface of the star. The drawback, of course, are the coordinate singularities appearing in the PDEs, but with

spectral methods those difficulties can be easily overcome if we introduce the angular potentials  $\eta$  and  $\mu$  defined as (Bonazzola & Marck 1990, Bonazzola *et al.* 1996)

$$B_\theta = \partial_\theta \eta - \frac{1}{\sin \theta} \partial_\phi \mu, \quad B_\phi = \frac{1}{\sin \theta} \partial_\phi \eta + \partial_\theta \mu \quad (14)$$

or analogously

$$\begin{aligned} \Delta_{\theta,\phi} \eta &= \frac{\partial^2 \eta}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial \eta}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial \eta}{\partial \phi^2} \\ &= \frac{\partial B_\theta}{\partial \theta} + \frac{\cos \theta}{\sin \theta} B_\theta + \frac{1}{\sin \theta} \frac{\partial B_\phi}{\partial \phi} \end{aligned} \quad (15)$$

Now, if we expand  $\eta$  in spherical harmonics we have

$$\Delta_{\theta,\phi} \eta = -l(l+1)\eta \quad (16)$$

Analogously for  $\mu$

$$\Delta_{\theta,\phi} \mu = \partial_\theta B_\phi + \frac{\cos \theta}{\sin \theta} B_\phi - \frac{1}{\sin \theta} \partial_\phi B_\theta \quad (17)$$

To illustrate the use of these potential by some examples, we shall mention that the divergence of a vector  $\vec{B}$  can be written

$$\text{div} \vec{B} = \partial_r B_r + \frac{2}{r} B_r + \frac{1}{r} \left( \partial_\theta B_\theta + \frac{\cos \theta}{\sin \theta} B_\theta + \frac{1}{\sin \theta} \partial_\phi B_\phi \right), \quad (18)$$

which reads, in spherical harmonic representation,

$$\partial_r B_r + \frac{2}{r} B_r - \frac{1}{r} l(l+1)\eta. \quad (19)$$

In the same way, to deal with an equation like  $\Delta \vec{B} = \vec{S}$  for a divergence-free vector  $\vec{B}$ , the decomposition in potentials of both  $\vec{B}$  and  $\vec{S}$  leads to solve

$$\frac{d^2 B_r}{dr^2} + \frac{4}{r} \frac{d}{dr} B_r + \frac{1}{r^2} (2 - l(l+1)) B_r = S_r, \quad (20)$$

$$\frac{d^2 \eta}{dr^2} + \frac{2}{r} \frac{d\eta}{dr} + \frac{1}{r^2} (-l(l+1)\eta + 2B_r) = \eta_S, \quad (21)$$

$$\frac{d^2 \mu}{dr^2} + \frac{2}{r} \frac{d\mu}{dr} - \frac{1}{r^2} l(l+1)\mu = \mu_S, \quad (22)$$

where Eq.(20) was obtained using Eq.(19) and the divergence-free nature of  $\vec{B}$ . Notice that the equation for the toroidal part  $\mu$  is decoupled from the other two equations.

Having obtained ordinary differential equations for the radial variable, we see that the problem of the singularities on the axis  $\theta = -\pi$ ,  $\theta = \pi$  is easily solved by means of spectral decomposition on spherical harmonics. Similarly, problems of singularities at  $r = 0$  are solved by expanding the solution on a complete set of functions that have a good analytical behaviour at  $r = 0$  (e.g. Chebyshev polynomials or a linear combination of them, see Appendix of Paper I).

### 3.3 Divergence-free decomposition

We shall now describe in more detail the new method that is used to apply Helmholtz theorem and decompose a vector field into a divergence-free and a potential parts. Consider again the equations of motion Eq. (12, 13), where  $S_i$  is a source term that is supposed to be known at the time  $t_{j-1}$ , and that we have to decompose into a divergence-free part  $\tilde{S}_i$  and a potential part  $\partial_i\Phi$ ,

$$S_i = \tilde{S}_i + \partial_i\Phi. \quad (23)$$

The ordinary way to make this decomposition is described in all analysis textbooks and was applied in Paper I: one simply takes the divergence of the Eq. (23) and solves the Poisson equation

$$\Delta\Phi = \text{div}S. \quad (24)$$

Once  $\Phi$  is obtained<sup>6</sup>, we got  $\tilde{S}_i$  by  $\tilde{S}_i = S_i - \partial_i\Phi$ .

The problem in the numerical implementation of this method is that in order to obtain  $\tilde{S}_i$  from  $\Phi$  one needs to compute its gradient, while the numerical computation of a derivative means higher round-off errors and generation of numerical instabilities. To avoid this problem, in the MHD code, we directly get  $\tilde{S}$  by first computing  $\mathcal{C}_\eta$ , the  $\eta$  angular potential of the curl of  $S$ , and then solving the system

$$(\text{curl}\vec{S})_\eta = (\text{curl}\vec{\tilde{S}})_\eta, \quad \text{div}\vec{\tilde{S}} = 0. \quad (25)$$

After an expansion in spherical harmonics, this system reads

$$\frac{d\eta}{dr} + \frac{\eta}{r} - \frac{\tilde{S}_r}{r} = \mathcal{C}_\eta, \quad \frac{d\tilde{S}_r}{dr} + \frac{2}{r}\tilde{S}_r - \frac{l(l+1)}{r}\eta = 0, \quad (26)$$

where  $\eta$  is the potential of  $\tilde{S}$ . An expansion in Chebyshev polynomials leads to an algebraic linear system whose matrix is a  $2(N_r - 1) \times (2N_r - 1)$  one ( $N_r$  being the number of coefficients in the expansion) that can be reduced easily to a 8-diagonals matrix, and in this way,  $\tilde{S}$  is solved without using  $\Phi$ .

### 3.4 Solution for the equation of a divergence-free vector

In the general case, the MHD equations are not linear and the solutions with zero viscosity (infinite Reynolds number) or/and zero conductivity (infinite magnetic Reynolds number) present discontinuity in the derivatives (infinite shear). Consequently, they cannot be described by numerical solutions with finite resolution and require specific techniques. Hence, in the following, we mainly consider the general case with both conductivity  $\sigma$  and viscosity  $\zeta$  not equal to zero, particular cases for which  $\zeta$  or/and  $\sigma$  vanish (*i.e.* linearized problems) being only partially discussed in the next Section, but mainly kept for another article. Furthermore, we describe here only the method used to solve the equation for the magnetic field  $\vec{B}$ , the case of the equation of the density momentum  $\vec{\mathcal{P}}$  being similar since the dispersive equation for the magnetic field  $B$  [Eq. (8)] can be written

$$\partial_t\vec{B} + 4\pi c^2 \text{curl}(\sigma \text{curl}\vec{B}) = S, \quad (27)$$

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<sup>6</sup>As already mentioned,  $\Phi$  is defined within a harmonic functions  $\Phi_A$  that is kept as an additional degree of freedom used for the implementation of BC.

where  $S$  contains all other terms.

To proceed, we use the decomposition of the vector  $\vec{B}$  in radial component  $B_r$  and two angular potential  $\eta$  and  $\mu$  as explained above (Eq. 14). The source term is calculated at time  $t_{j+1/2}$  using the previous values at time  $t_j$  and  $t_{j-1}$  by

$$S^{j+1/2} = \frac{3S^j - S^{j-1}}{2}. \quad (28)$$

For the moment, we will consider  $\sigma$  constant, the non-constant case being technically identical (see Appendix of Paper I). Using the identity

$$\text{curl curl } \vec{B} = -\Delta \vec{B} + \overrightarrow{\text{grad}}(\text{div} \vec{B})$$

and taking into account the divergence-free nature of  $\vec{B}$ , an expansion in spherical harmonics gives the system [see Eqs. (20,21,22)]

$$\partial_t B_r + \sigma \left[ \frac{d^2 B_r}{dr^2} + \frac{4}{r} \frac{dB_r}{dr} + \frac{1}{r^2} (2 - l(l+1)) B_r \right] = S_r \quad (29)$$

$$\partial_t \eta + \sigma \left[ \frac{d^2 \eta}{dr^2} + \frac{2}{r} \frac{d\eta}{dr} + \frac{1}{r^2} (-l(l+1)\eta + 2B_r) \right] = S_\eta \quad (30)$$

$$\partial_t \mu + \sigma \left[ \frac{d^2 \mu}{dr^2} + \frac{2}{r} \frac{d\mu}{dr} - \frac{1}{r^2} l(l+1)\mu \right] = S_\mu \quad (31)$$

where  $S_\eta$  and  $S_\mu$  are respectively the angular potentials of the source  $S_i$ . Remember that the condition  $\text{div} \vec{B} = 0$  reads

$$\frac{dB_r}{dr} + \frac{2}{r} B_r - \frac{1}{r} l(l+1)\eta = 0. \quad (32)$$

Notice also that the equations for  $B_r$  and  $\mu$  are decoupled and behave like the Poisson equation.

There are different strategies to solve the above system: we can, for example solve the Eq. (29) and compute  $\eta$  by using the divergence [Eq. (32)]:

$$\eta = \frac{1}{l(l+1)} \left( r \frac{dB_r}{dr} + 2B_r \right).$$

This method is very simple, but its drawback is that we have to perform a derivative with respect to  $r$  of  $B_r$ , that generates high spacial frequency noise, which in turn produces numerical instabilities in a MHD code. We can also use the method described in Paper I, but we present here a different one that is more robust.

This new method is the following: After the time discretisation, a second order (in time) implicit scheme consists in re-writing the Eqs. (29, 30) in the following way:

$$B_r^{j+1} - \frac{\delta t}{2} \sigma \left[ \frac{d^2 B_r^{j+1}}{dr^2} + \frac{4}{r} \frac{dB_r^{j+1}}{dr} + \frac{1}{r^2} (2 - l(l+1)) B_r^{j+1} \right] = B_r^j + \frac{\delta t}{2} \sigma \left[ \frac{d^2 B_r^j}{dr^2} + \frac{4}{r} \frac{dB_r^j}{dr} + \frac{1}{r^2} (2 - l(l+1)) B_r^j \right] + \delta t S_r^{j+1/2}, \quad (33)$$

$$\eta^{j+1} - \frac{\delta t}{2} \sigma \left[ \frac{d^2 \eta^{j+1}}{dr^2} + \frac{2}{r} \frac{d\eta^{j+1}}{dr} + \frac{1}{r^2} (2B_r^{j+1} - l(l+1)\eta^{j+1}) \right] = \eta^j + \frac{\delta t}{2} \sigma \left[ \frac{d^2 \eta^j}{dr^2} + \frac{2}{r} \frac{d\eta^j}{dr} + \frac{1}{r^2} (2B_r^j - l(l+1)\eta^j) \right] + \delta t S_\eta^{j+1/2}, \quad (34)$$



where  $\delta t$  is the time increment and  $S^{j+1/2}$  is the value of the source extrapolated at time  $t_j + \delta t/2$  as explained by Eq.(28). Note the presence of the singular term  $(2B_r^j - l(l+1)\eta^j)/r^2$  at the RHS of the Eq. (34). An elementary study of the analytical properties of the above system of equations shows that the solutions vanish at least as  $r^{l-1}$  at the origin  $r = 0$ .<sup>7</sup> Consequently for  $l \geq 3$ , the solutions vanish fast enough to avoid actual singularities in the apparently singular terms of Eq. (34) RHS. For  $l = 1, 2$ , those terms are indeed singular, but the solution is not singular because they do compensate exactly. However, in order to overcome this numerical difficulty, we re-write Eq. (34) by using Eq. (32) as follows:

$$\begin{aligned} \eta^{j+1} - \frac{\delta t}{2} \sigma \left[ \frac{d^2 \eta^{j+1}}{dr^2} + \frac{2}{r} \frac{d\eta^{j+1}}{dr} - \frac{1}{r^2} (l(l-1)\eta^{j+1}) \right] = \\ \eta^j + \frac{\delta t}{2} \sigma \left[ \frac{d^2 \eta^j}{dr^2} + \frac{2}{r} \frac{d\eta^j}{dr} - \frac{1}{r^2} l(l-1)\eta^j \right] + (S^1 + S_\eta^{j+1/2}) \delta t, \end{aligned} \quad (35)$$

where  $S^1$  is a new source term:

$$S^1 = \frac{\sigma}{2r^2} \left[ -\frac{2}{l+1} \left( r \frac{dB_r^{j+1/2}}{dr} + 2B_r^{j+1/2} \right) + 2B_r^{j+1/2} \right] \quad (36)$$

It is easy to see that for  $l = 1$  or  $l = 2$ , since  $B_r^l = b_l r^{l-1} + \dots$  and  $\eta^l = e_l r^{l-1} + \dots$ , all the terms in Eq. (35) are regular.

Then, the way to proceed consists in first, getting homogeneous solutions by doing the following:

- a) - Compute an homogeneous solution of the system of Eq. (33, 34): solve with  $S_r^{j+1/2} = 0$  for calculating  $B_r^{j+1}$ , let  $B_r^h$  be this solution.
- b) - Compute an homogeneous solution  $\eta_h$  of the Eq. (35): solve with  $S^1 = 0$  and  $S_\eta^{j+1/2} = 0$
- c) - Compute a particular solution  $\eta^{h1}$  of the Eq. (35) using  $S^{j+1/2} = 0$  and  $S^1$ , the latter being computed from  $B_r^h$
- d) - Impose, by using the homogeneous solution  $\eta_h$ , that the divergence given by the Eq. (32) vanishes at the boundary ( $r = R$ ).

In such a way, we have a homogeneous solution of the system (33, 34) that satisfies exactly the condition of vanishing divergence ( $S_r^{j+1/2} = 0$ ,  $S_\eta^{j+1/2} = 0$ ) at the boundary. All those homogeneous solutions are stored and have to be computed at each time-step whether  $\delta t \sigma$  depends on time. The complete solution is computed by obtaining, in a similar way, a particular solution and by using the homogeneous solution to satisfy the BC. The condition  $\text{div} \vec{B} = 0$ , as was already said, is fulfilled exactly by construction at the boundary and approximately in the interval  $0 \leq r \leq R$ . For a given number of degrees of freedom  $N_r$  in  $r$ , the error  $\epsilon$  depends on the value of  $\delta t \sigma$ . In practical cases, we had  $10^{-11} \leq \epsilon \leq 10^{-5}$ .

## 4 Non-dissipative MHD

In the previous section, we have discussed the general MHD problem (non-vanishing viscosity and resistivity). As explained, in this case, the

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<sup>7</sup>The value  $l = 0$  is forbidden, because we treat a pure vector field.

presence of viscosity and resistivity allows us to easily impose BCs and help to stabilize. Yet, there are astrophysical steady-state problems for which we really need a non-dissipative code, for example while probing the stability of hydromagnetic configurations. For such a class of problems, linearized MHD has to be used, viscosity and resistivity being no more necessary to smooth the solutions, being on the contrary to be avoided. Hence, the implementation of BCs has to be done in a different way. This technology is in progress, and as a first illustration, we present here an example of application inspired by a model of  $\gamma$ -ray bursts proposed recently by Paczyński & Haensel (2005) and based on previous proposals by Kluźniak & Ruderman (1998) and Dai & Lu (1998).

This model consists in a just born differentially rotating and magnetized NS, that during the cooling time (and consequently the shrinking time) undergoes a phase transition and becomes a *differentially rotating stratified quark star*. At this point, the poloidal magnetic field is stretched by the differential rotation and a fraction of the rotational kinetic energy related with the differential rotation is transformed into magnetic energy of a toroidal field. Orders of magnitude estimation shows that for a star initially rotating at 200 Hz, with an amount of some few percent of the total rotational kinetic energy due to the differential rotation, energy budget is met. Moreover, a strong toroidal magnetic field ( $\sim 10^{15}$  gauss) is generated. If there are no losses, the magnetic field reaches a maximum and starts to decrease, transforming its magnetic energy back into differential kinetic energy (at constant kinetic momentum). The duration of this twisting period depends on the intensity of the preexisting poloidal magnetic counterpart  $\vec{B}_p$  ( $\sim 100$  rotation periods for  $B_p = 10^{13}$  gauss). From a numerical point of view, the winding time of the toroidal magnetic field  $B_{phi}$ , takes about hundred rotation periods of the star, meaning that it is crucial to avoid important energy losses.

The low dissipation condition cannot be fulfilled with the code described above reasonable numerical viscosity and resistivity. Hence, we modified it, taking into account the particularity of the problem in order to eliminate numerical dissipation. The strong stratification of the star suggests the approximation  $V_r = 0$  and  $V_\theta = 0$ , since motions of the matter inside the star tend to be parallel to isopotential surfaces:  $V_r \sim 0$ ,  $V_\theta \sim 0$ . Consequently, only evolutions of the toroidal components of the magnetic field  $B_\phi$  and of the velocity  $V_\phi$  are pertinent to our problem. Moreover, as a first step, we consider in the following an incompressible fluid and Newtonian gravity.

#### 4.1 Evolution of the toroidal component $B_\phi$

The equations of motion for the toy model are

$$\partial_t \vec{B} = \text{curl}(\vec{V} \wedge \vec{B}), \quad (37)$$

and

$$\partial_t \vec{V} = \frac{1}{4\pi n} (\text{curl} \vec{B}) \wedge \vec{B}. \quad (38)$$

Taking into account the assumptions described above, the axisymmetry and the equatorial symmetry, the equations to be solved read

$$\partial_t B_\phi - B_r \left( \partial_r V_\phi - \frac{1}{r} V_\phi \right) - \frac{B_\theta}{r} \left( \partial_\theta V_\phi - \frac{1}{r} \frac{\cos \theta}{\sin \theta} V_\phi \right) = 0, \quad (39)$$

$$\partial_t V_\phi - \frac{1}{4\pi n} \left[ B_r \left( \partial_r B_\phi + \frac{1}{r} B_\phi \right) + \frac{B_\theta}{r} \left( \partial_\theta B_\phi + \frac{\cos \theta}{\sin \theta} B_\phi \right) \right] = 0, \quad (40)$$

from which we see that in the case of rigid rotation,  $V_\phi = \Omega r \sin \theta$ ,  $B_\phi$  does not evolve.

In what follows, we shall assume in addition that the poloidal magnetic field can be written (taking into account the symmetries of the problem)

$$B_r = \cos \theta (b_1 + b_2 r^2 \cos(2\theta) + \dots),$$

$B_\theta$  being obtained by the condition  $\text{div} \vec{B} = 0$ .

We have to impose BCs in order to have an unique solution. A study of Eqs.(39) and (40) shows that this system is hyperbolic, and consequently, one BC can be imposed at the surface of the star. Physically, what occurs is that a radial Alfvén wave is created in order to satisfy the given BC. Since in the local approximation, the phase velocity  $V_{Al}$  of an Alfvén wave verifies

$$V_{Al}^2 = \frac{w^2}{k^2} = \frac{(\vec{k} \cdot \vec{B})^2}{4\pi n k^2}, \quad (41)$$

for an Alfvén wave propagating radially, the phase velocity is

$$V_{Al}^2 = \frac{1}{4\pi n} B_r^2, \quad (42)$$

which is maximum at the pole ( $\theta = 0$ ) and vanishes for  $\theta = \pi/2$  (the equator), meaning some possible numerical troubles due to the large variation of the timescale.

Hence, the technique to solve this system is to work in configuration space in  $\theta$  and in the Chebyshev space for  $r$ . Once the discretisation of  $\theta$  is done, we are confronted with  $N_\theta$  hyperbolic systems that can be solved implicitly. Since the Alfvén velocity can be very small for  $\theta$  close to  $\pi/2$ , the number of degrees of freedom  $N_r$  for  $r$  must be high enough to describe the Alfvén wave that propagates very slowly along the radius. In conclusion,  $N_r$  depends on  $N_\theta$ . Note that for  $\theta = \pi/2$  (the equator) the system is completely degenerated, and  $B_\phi$  vanishes as it is anti-symmetric with respect to the equator.

To conclude, we emphasize that the finale solution of the system strongly depends on the chosen BCs. For example, if we choose  $B_\phi = 0$   $|_{r=R}$ , the Poynting vector vanishes at the surface, and the energy is conserved<sup>8</sup>. Thus, in the next Section, we shall provide some examples with conditions that allows us to easily check the energy conservation and consequently to test the accuracy of the code. More precisely, we shall use for the non-dissipative case the condition  $B_r = 0$   $|_{r=R}$  at  $t = 0$ , which makes the operator degenerated at the surface, implying that no BC is required but the Poynting vector does vanish at the surface.

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<sup>8</sup>Note that the radial component  $J_r$  of the current vanishes at the surface too:

$$J_r = \frac{1}{r} (\partial_\theta B_\phi + \frac{\cos \theta}{\sin \theta} B_\phi).$$

## 5 Tests

In the following, we shall illustrate the stability of the code, but mainly with tests that concern linear (or linearised) problems. Yet, we would like to point out again that linear problems, when using spectral methods, are the most severe ones. Indeed, for a linear problem, a spectral code has to work with vanishing viscosity and conductivity, and since spectral methods have no intrinsic numerical dissipation, numerical instabilities have greater opportunity to grow than in the non-linear case, where viscosity and resistivity can be increased until the code becomes stable. This is the reason why the stability conditions illustrated in the following were obtained in a non-trivial way, that will nevertheless be explained in detail somewhere else.

### 5.1 Dissipative case

Since the dissipative case is far being the most interesting one from the numerical point of view, we only provide one example of evolution in the case of a hydrodynamical coupling between an r-mode (lower frequency) and  $l = 2$  f-mode (higher frequency). The calculation pictured on Fig.1 shows the time evolution of the component  $v_\phi$  of the velocity on the surface of the star. The calculation is done with the number of coefficient in spectral expansion equal to 24, 6 and 4 in  $r$ ,  $\theta$  and  $\phi$  directions, respectively, and the time-step is  $10^{-2}$  (in the computational units in which the period of rotation is equal unity). The dimensionless viscosity parameter equals 0.001.

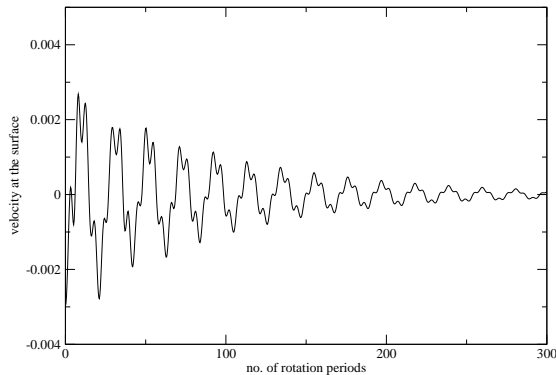


Figure 1: Time evolution of the velocity  $v_\phi$  of a given point on the surface of the star for a dissipative code with excited  $l = 2$  r- and f-modes.

### 5.2 Non-dissipative case

We also present a couple of illustrative tests of the spectral code, that were performed with the linear version, neglecting all dissipative terms. On Figs. 2 and 3, the case of a pure hydrodynamical coupling between an r-mode and  $l = 2$  f-mode is shown. As in previous subsection, the number of coefficient in spectral expansion is again 24, 6 and 4 in  $r$ ,  $\theta$  and  $\phi$  directions, respectively, while the time-step is  $10^{-2}$ .

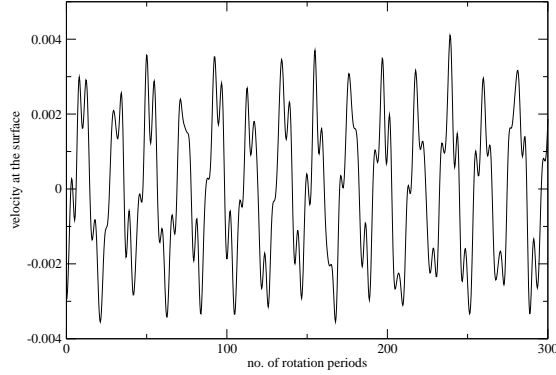


Figure 2:  $v_\phi$  component of the velocity for a given point on the surface of the star with excited  $l = 2$  r- and f-modes, versus time.

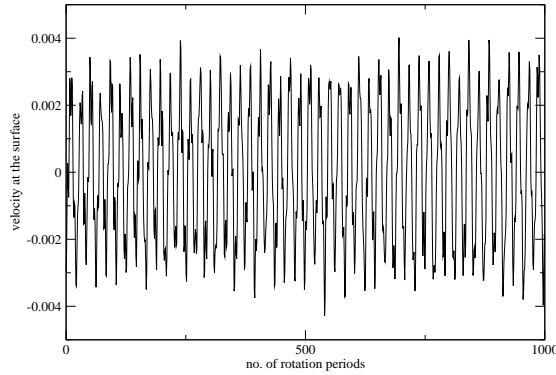


Figure 3: The same as on Fig. 2, but for a longer run, in order to illustrate the lack of dissipation and the long-term stability of the code.

Last but not least, we shall also plot the conservation of energy in a linear MHD code using the no-radial-magnetic field condition (see end of previous Section) and no dissipation term. The chosen model is axisymmetric (24 coefficients in  $r$ -direction and 16 in  $\theta$ -direction) with the time-step equal  $10^{-3}$  and with an initial  $v_\phi$  profile  $r \sin \theta - r^3 \sin^3 \theta$  and an initial magnetic field with only a radial component:  $B_r = \sin \theta (1 - (r/R)^2)$ . On Fig. 4, it can be seen that for 2.5 rotation periods (2500 time-steps), energy is conserved up to the value of  $10^{-8}$  (the difference between the total initial and final energy), meaning a relative variation around  $10^{-5}$  only. As expected, at the end of the cycle, the star has gained strong differential rotation and a poloidal component of the magnetic field.

## 6 Conclusions

We have described a code based on spectral methods able to handle 3D relativistic MHD problems in a sphere (e.g. the interior of a slowly rotating

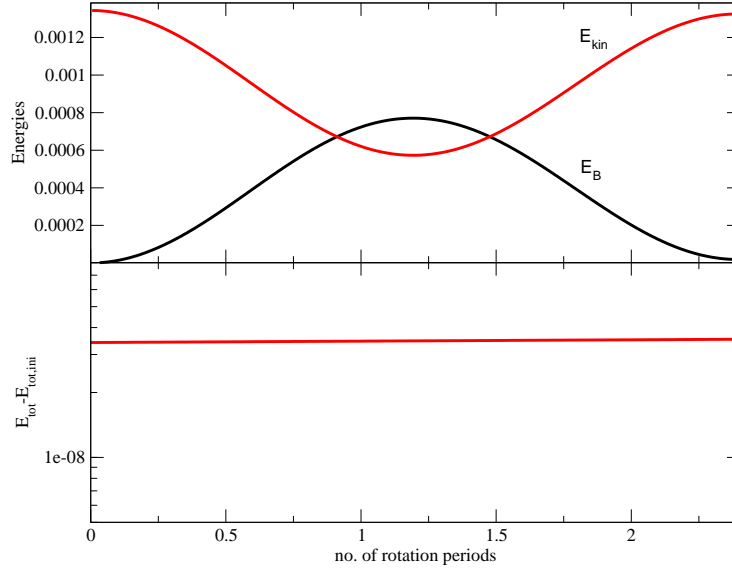


Figure 4: The interplay between the kinetic energy  $E_{\text{kin}}$  and magnetic energy  $E_{\text{B}}$  as well as the variation of the total energy during the creation of a toroidal component of the magnetic field from an initial purely poloidal magnetic field, due to differential rotation (see text for details).

neutron star). The code works with spherical coordinates and spherical components of the velocity  $\vec{V}$  and of the magnetic field  $\vec{B}$ . The use of spherical coordinates and spherical components allows us to impose exact BCs on the surface of the star. Moreover, thanks to spectral methods, the singularities present on the  $z$  axis and at  $r = 0$ , which result from spherical coordinates, can be treated properly. We want to point out that the numerical solution is obtained in the *whole* star, while numerous dynamical spectral codes (for instance in geophysics) are restricted to some shells.

As far as physics is concerned, order of magnitude considerations show that for most of the known compact stars, the acoustic time scale is much shorter than other timescales (Alfvén, Brunt-Väisälä, rotation timescale, ...). In order to easily have an efficient code, the anelastic approximation, which filters acoustic waves, is employed. In such a way, slow evolution of the star can be followed during tenths of its rotations. Moreover, due to the intrinsic accuracy and low dissipativity of spectral methods, the code is well suited to study the stability of steady-state MHD configurations and other physical problems that we shall investigate elsewhere.

## Acknowledgements

One of us (SB) thanks Dr. J. Novak for helpful discussions. LV and MB were supported, respectively, by the Marie Curie Intra-european Fellowships MEIF-CT-2005-025498 and MEIF-CT-2005-023644, within the 6th European Community Framework Programme.

## References

- Antón, L., Anett, O., Miralles, J.A., Martí, J.M., Ibáñez, J.M., Font, J.A., Pons, J.A., ApJ **637**, 296 (2006)
- Bonazzola, S., Friebe, J., Gourgoulhon, E. Astrophys. J. **3012**,675 (1996)
- Bonazzola, S. Marck, J.A. J.Comp. Phys. **87**,201 (1990)
- Bucciantini, N., Thompson, T.A., Arons, J., Quataert, E., Del Zanna, L., Mon. Not. Roy. Astron. Soc. **368**, 1717 (2006)
- Dai, Z.G., Lu, T., Phys. Rev. Let. **81**, 4301 (1998)
- De Villiers, J.-P. Hawley, J. F., ApJ **589**, 458 (2003)
- Duez, M. D., Liu, Y. T., Shapiro, S. L., Stephens, B. C., Phys. Rev. D **72**, 024028 (2005)
- Gammie, J.C., McKinney, J.C., Tóth, G., ApJ **589**, 444 (2003)
- Goldreich, P., Reisenegger, A., ApJ **395**, 250 (1992)
- Kluźniak, W., Ruderman, M., ApJ **505**, L113 (1998)
- Koide, S., Shibata, K., Kudoh, T., ApJ **522**, 727 (1999)
- Komissarov, S.S., Mon. Not. Roy. Astron. Soc. **350**, 1431 (2004)
- Lichnerowicz, A., Relativistic Hydrodynamics and Magnetohydrodynamics, New York: Benjamin, (1967)
- Mizuno, Y., Nishikawa, K.I., Koide, S., Hardee, P., Fishman, G.J., astro-ph/0609004
- Paczynski, B., Haensel, P. Mon. Not. Roy. Astron. Soc. Lett. **362**, L4 (2005)
- Shibata, M., Sekiguchi, Y.-I., Phys. Rev. D **72**, 044014 (2005)
- Villain, L., Bonazzola, S. Phys. Rev. D **66**, 123001 (2002)
- Villain, L., Bonazzola, S., Haensel, P. Phys.Rev. D **71**, 083001 (2005)
- Wilson, J. R. ApJ **172**, 431 (1972)